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On the equivalence of sources and duality of fields in isotropic chiral media

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Abstract. The derivation of various theorems dealing with the equivalence of electric and magnetic sources and the duality of the radiated fields in isotropic chiral media ($\boldsymbol{D}=\varepsilon\boldsymbol{E}+\beta\varepsilon\nabla\times\boldsymbol{E}$, $\boldsymbol{B}=\mu\boldsymbol{H}+\beta\mu\nabla\times\boldsymbol{H}$) is given.

1. Introduction

Linearly polarised waves cannot propagate through chiral media. When an electromagnetic disturbance travels through such a medium, it is forced to adapt to the handedness of the microstructure. In other words, left- and right-circularly polarised plane waves, travelling with different phase velocities, are perfectly acceptable for this class of media [1].

In order to describe the electromagnetic properties of *isotropic* chiral media, the usual constitutive equations, $\boldsymbol{D}=\varepsilon\boldsymbol{E}$ and $\boldsymbol{B}=\mu\boldsymbol{H}$, are inadequate because they admit to a single phase velocity. Instead,

$$\boldsymbol{D}=\varepsilon[\boldsymbol{E}+\beta\nabla\times\boldsymbol{E}] \tag{1a}$$

$$\boldsymbol{B}=\mu[\boldsymbol{H}+\beta\nabla\times\boldsymbol{H}] \tag{1b}$$

which constitutive relations are symmetric under time reversality [2]; it will be assumed hereafter in this paper that ε and μ may be complex, but β is real. For a right-handed medium $\beta>0$ while, for a left-handed medium, $\beta<0$; β, of course, is the measure of chirality, and it equals zero for achiral materials.

The various aspects of the electromagnetic field theory applicable to isotropic chiral media have recently begun to be explored [3-6]. The authors have elsewhere [7] given the *infinite medium* Green dyadic [8]† as well as derived the Huyghens principle for the electric and the magnetic fields in isotropic chiral media, and employed them to set up and investigate a pertinent scattering formalism. As part of their ongoing efforts to understand the interaction of electromagnetic waves with chiral media, the authors report here the derivation of various theorems dealing with the equivalence of electric and magnetic sources and the duality of the radiated fields.

The monochromatic Maxwell's equations with an $\exp[-\mathrm{i}\omega t]$ time dependence

$$\nabla\times\boldsymbol{E}=\mathrm{i}\omega\boldsymbol{B}-\boldsymbol{K} \tag{2a}$$

$$\nabla\times\boldsymbol{H}=-\mathrm{i}\omega\boldsymbol{D}+\boldsymbol{J} \tag{2b}$$

† These authors have derived the Green dyadic for the constitutive equations $\boldsymbol{D}=\varepsilon_{\mathrm{P}}\boldsymbol{E}+\mathrm{i}\beta_{\mathrm{P}}\boldsymbol{B}$, $\boldsymbol{B}=\mu_{\mathrm{P}}\boldsymbol{H}-\mathrm{i}\mu_{\mathrm{P}}\beta_{\mathrm{P}}\boldsymbol{E}$. By making use of the mapping $\varepsilon_{\mathrm{P}}=\varepsilon$, $\beta_{\mathrm{P}}=\omega\varepsilon\beta$ and $\mu_{\mathrm{P}}=\mu/(1-\omega^2\varepsilon\mu\beta^2)$ the Green dyadic for the constitutive equations (1a, b) can be derived and is given in 5(a, b, c).

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can be simplified by the elimination of $\boldsymbol{D}$ and $\boldsymbol{B}$ so that

$$(1-k^2\beta^2)\nabla\times\boldsymbol{E}=\mathrm{i}\omega\mu\boldsymbol{H}+k^2\beta\boldsymbol{E}-\boldsymbol{K}+\mathrm{i}\omega\mu\beta\boldsymbol{J} \tag{3a}$$

$$(1-k^2\beta^2)\nabla\times\boldsymbol{H}=-\mathrm{i}\omega\varepsilon\boldsymbol{E}+k^2\beta\boldsymbol{H}+\boldsymbol{J}+\mathrm{i}\omega\varepsilon\beta\boldsymbol{K}. \tag{3b}$$

$\boldsymbol{J}$ and $\boldsymbol{K}$, respectively, are the radiating electric and magnetic current densities. Parenthetically, it is mentioned here that though the magnetic sources are completely fictitious, their use is a time-honoured device to convert 'difficult' problems involving the electric sources into 'simpler' ones involving magnetic sources [9]. Furthermore, on dielectric-dielectric interfaces, it is common practice to consider the establishment of 'equivalent' sources, $\boldsymbol{J}_s=\boldsymbol{e}_n\times\boldsymbol{H}_s$ and $\boldsymbol{K}_s=\boldsymbol{E}_s\times\boldsymbol{e}_n$, $\boldsymbol{e}_n$ being a unit normal to the interface and $\boldsymbol{E}_s$ and $\boldsymbol{H}_s$ being the actual fields on the interface; this equivalence is extensively used in the T-matrix method [10] as well as in the method of moments [11].

With further manipulation of (3), along with use of the constitutive equations (1), it has been shown [7] that the radiated fields satisfy the governing differential equations

$$\mathfrak{D}(\boldsymbol{r})\cdot\boldsymbol{E}=\mathrm{i}\omega\mu(\gamma/k)^2[\boldsymbol{J}+\beta\nabla\times\boldsymbol{J}]-(\gamma/k)^2[\nabla\times\boldsymbol{K}] \tag{4a}$$

$$\mathfrak{D}(\boldsymbol{r})\cdot\boldsymbol{H}=\mathrm{i}\omega\varepsilon(\gamma/k)^2[\boldsymbol{K}+\beta\nabla\times\boldsymbol{K}]+(\gamma/k)^2[\nabla\times\boldsymbol{J}] \tag{4b}$$

in which the dyadic differential operator $\mathfrak{D}(\boldsymbol{r})$ is given by

$$\mathfrak{D}(\boldsymbol{r})=\nabla\times\nabla\times\mathfrak{J}-2\gamma^2\beta\nabla\times\mathfrak{J}-\gamma^2\mathfrak{J} \tag{4c}$$

with $\mathfrak{J}$ being the identity dyadic. In the preceding equations, $k=\omega[\varepsilon\mu]^{1/2}$ is merely a shorthand notation, and $\gamma^2/k^2=[1-k^2\beta^2]^{-1}$. Pertinent to the constitutive equations (1*a*, *b*), the infinite medium Green dyadic is given as [7, 8]

$$\mathfrak{G}(\boldsymbol{r},\boldsymbol{r}_0)=\mathfrak{G}_1(\boldsymbol{r},\boldsymbol{r}_0)+\mathfrak{G}_2(\boldsymbol{r},\boldsymbol{r}_0) \tag{5a}$$

where

$$\mathfrak{G}_1(\boldsymbol{r},\boldsymbol{r}_0)=(k/8\pi\gamma^2)[\gamma_1\mathfrak{J}+\nabla\nabla/\gamma_1+\nabla\times\mathfrak{J}]g(\gamma_1;R) \tag{5b}$$

$$\mathfrak{G}_2(\boldsymbol{r},\boldsymbol{r}_0)=(k/8\pi\gamma^2)[\gamma_2\mathfrak{J}+\nabla\nabla/\gamma_2-\nabla\times\mathfrak{J}]g(\gamma_2;R) \tag{5c}$$

with $\gamma_1=k[1-k\beta]^{-1}$, $\gamma_2=k[1+k\beta]^{-1}$, $g(\kappa;R)=\exp[\mathrm{i}\kappa R]/R$ and $\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}_0$.

2. Equivalence of current sources

It is often possible that a field problem is simplified by replacing electric sources with equivalent magnetic sources and vice versa [9]. To that end, for homogeneous, achiral, isotropic media the necessary source-equivalence theorems were derived by Mayes [12]. In this section, similar theorems are derived for isotropic chiral media.

Theorem 1. Let $\{\boldsymbol{E}_1,\boldsymbol{H}_1\}$ be the fields produced by an electric current density $\boldsymbol{J}$, while a magnetic current distribution $\boldsymbol{K}$ independently creates the fields $\{\boldsymbol{E}_2,\boldsymbol{H}_2\}$. If the two source current distributions $\boldsymbol{J}$ and $\boldsymbol{K}$ are such that

$$\boldsymbol{J}+\beta\nabla\times\boldsymbol{J}=-\nabla\times\boldsymbol{K}/\mathrm{i}\omega\mu \tag{6}$$

then they produce identical electric fields, i.e. $\boldsymbol{E}_1=\boldsymbol{E}_2$, while $\boldsymbol{B}_1-\boldsymbol{B}_2=-\boldsymbol{K}/\mathrm{i}\omega$.

Proof. From (4) it is easy to see that

$$\mathfrak{D}\cdot(\boldsymbol{E}_1-\boldsymbol{E}_2)=(\gamma/k)^2(\mathrm{i}\omega\mu[\boldsymbol{J}+\beta\nabla\times\boldsymbol{J}]+\nabla\times\boldsymbol{K}) \tag{7a}$$

$$\mathfrak{D}\cdot(\boldsymbol{H}_1-\boldsymbol{H}_2)=(\gamma/k)^2(\nabla\times\boldsymbol{J}-\mathrm{i}\omega\varepsilon[\boldsymbol{K}+\beta\nabla\times\boldsymbol{K}]) \tag{7b}$$

which yield the identity $E_1 = E_2$ provided (6) holds. At the same time, from (3*b*) one can obtain

$$\nabla \times (H_1 - H_2) = \gamma^2 \beta (H_1 - H_2) + (\gamma/k)^2 [J - i\omega\varepsilon\beta K] \tag{8a}$$

by enforcing the provision that $E_1 - E_2$; from (8*a*), therefore,

$$\nabla \times \nabla \times (H_1 - H_2) = \gamma^2 \beta \nabla \times (H_1 - H_2) + (\gamma/k)^2 [\nabla \times J - i\omega\varepsilon\beta \nabla \times K]. \tag{8b}$$

Substitution of (8*b*) in (7*b*), and the subsequent use of (1*b*), then yields the identity $B_1 - B_2 = -K/i\omega$.

It should be noted here that the specification of an E-equivalent K for a specified J is non-unique to the extent that $K \to K + \nabla\zeta$, $\zeta(r)$ being any arbitrary scalar field. This non-uniqueness of K does not affect the computation of the radiated electric field. From (4) and (5)

$$E_2(r) = -(\gamma/k)^2 \iiint d^3 r_0 \mathfrak{G}(r, r_0) \cdot [\nabla_0 \times K(r_0)] \tag{9}$$

which can in no way be influenced by the replacement of K by $K + \nabla\zeta$; r and r_0, respectively, are the field and the source points with r lying outside the source-carrying volume. It follows then also that $D_1(r) = D_2(r)$. On the other hand, the differences $B_1(r) - B_2(r)$ and $H_1(r) - H_2(r)$ are not unique (and not zero), but that consideration is not required of theorem 1.

Proceeding in the same way as for theorem 1, and with similar considerations, it can be shown that the following theorem also holds.

Theorem 2. Let $\{E_1, H_1\}$ be the fields produced by an electric current density J, while a magnetic current distribution K independently creates the fields $\{E_2, H_2\}$. If the two source current distributions J and K are such that

$$K + \beta \nabla \times K = \nabla \times J / i\omega\varepsilon \tag{10}$$

then they produce identical magnetic fields, i.e. $H_1 = H_2$, while $D_1 - D_2 = J/i\omega$. This prescription of an H-equivalent J is non-unique to the extent that $J \to J + \nabla\zeta$, $\zeta(r)$ being any arbitrary scalar field.

Of great interest would be finding J and K such that both conditions (6) and (10) are simultaneously satisfied, i.e. $E_1 = E_2$ along with $H_1 = H_2$. The concurrent solution of these conditions leads to the following.

Theorem 3. If there exists an electric current density J such that

$$\mathfrak{D} \cdot J = 0 \tag{11a}$$

then there also exists a magnetic current density K, given by

$$K = k^2 (i\omega\varepsilon)^{-1} [\nabla \times J/\gamma^2 - \beta J] \tag{11b}$$

such that they both produce the same electric and magnetic fields.

Proof. By substituting for $\beta\nabla \times K$ from (10) into (6), it is easy to see that $i\omega\varepsilon K = (k/\gamma)^2 \nabla \times J - k^2 \beta J$. Next, the curl of this equation is taken, and $\nabla \times K$ from (6) is substituted. As a result, $\mathfrak{D} \cdot \boldsymbol{J} = 0$. Furthermore, by substituting for $\nabla \times \boldsymbol{K}$ from (6) into (10), it can be seen that $\boldsymbol{K} = k^2(\mathrm{i}\omega\varepsilon)^{-1}[\nabla \times \boldsymbol{J}/\gamma^2 - \beta\boldsymbol{J}]$. Thus, the $\boldsymbol{J}$ and $\boldsymbol{K}$ given by (11*a*, *b*) simultaneously satisfy the constraints of both theorems 1 and 2.

Next, to show that these $\boldsymbol{J}$ and $\boldsymbol{K}$ produce the same fields, consider (4*a*) and (11*b*), whence

$$\mathfrak{D} \cdot \boldsymbol{E}_2 = -(\gamma/k)^2\nabla \times \boldsymbol{K} = -(\mathrm{i}\omega\varepsilon)^{-1}[\nabla \times \nabla \times \boldsymbol{J} - \gamma^2\beta\nabla \times \boldsymbol{J}]. \tag{12a}$$

On using now (10*a*), this can be simplified to

$$\mathfrak{D} \cdot \boldsymbol{E}_2 = (\mathrm{i}\omega\mu)(\gamma/k)^2[\boldsymbol{J} + \beta\nabla \times \boldsymbol{J}] = \mathfrak{D} \cdot \boldsymbol{E}_1. \tag{12b}$$

Likewise, from (4*b*) and (11), it can be shown that

$$\mathfrak{D} \cdot \boldsymbol{H}_2 = (\mathrm{i}\omega\varepsilon)(\gamma/k)^2[\boldsymbol{K} + \beta\nabla \times \boldsymbol{K}] = (\gamma/k)^2[\nabla \times \boldsymbol{J}] = \mathfrak{D} \cdot \boldsymbol{H}_1. \tag{13}$$

Finally comes the question of the integrity of the fields $\{\boldsymbol{E}_1, \boldsymbol{H}_1\}$ radiated by $\boldsymbol{J}$. Then, after using the Green dyadic (5), at a field point $\boldsymbol{r}$ where $\boldsymbol{J}(\boldsymbol{r}) = 0$, from (4) one has

$$\boldsymbol{E}_1(\boldsymbol{r}) = (\mathrm{i}\omega\mu)(\gamma/k)^2 \iiint \mathrm{d}^3\boldsymbol{r}_0\, \mathfrak{G}(\boldsymbol{r} - \boldsymbol{r}_0) \cdot [\boldsymbol{J}(\boldsymbol{r}_0) + \beta\nabla_0 \times \boldsymbol{J}(\boldsymbol{r}_0)] \tag{14a}$$

$$\nabla \times \boldsymbol{E}_1(\boldsymbol{r}) = (\mathrm{i}\omega\mu)(\gamma/k)^2 \iiint \mathrm{d}^3\boldsymbol{r}_0\, \mathfrak{G}(\boldsymbol{r} - \boldsymbol{r}_0) \cdot [\nabla_0 \times \boldsymbol{J}(\boldsymbol{r}_0) + \beta\nabla_0 \times \nabla_0 \times \boldsymbol{J}(\boldsymbol{r}_0)] \tag{14b}$$

$$\boldsymbol{H}_1(\boldsymbol{r}) = (\gamma/k)^2 \iiint \mathrm{d}^3\boldsymbol{r}_0\, \mathfrak{G}(\boldsymbol{r} - \boldsymbol{r}_0) \cdot [\nabla_0 \times \boldsymbol{J}(\boldsymbol{r}_0)] \tag{14c}$$

$$\nabla \times \boldsymbol{H}_1(\boldsymbol{r}) = (\gamma/k)^2 \iiint \mathrm{d}^3\boldsymbol{r}_0\, \mathfrak{G}(\boldsymbol{r} - \boldsymbol{r}_0) \cdot [\nabla_0 \times \nabla_0 \times \boldsymbol{J}(\boldsymbol{r}_0)]. \tag{14d}$$

Together these relations imply that

$$\begin{aligned}\nabla \times \boldsymbol{E}_1(\boldsymbol{r}) - \mathrm{i}\omega\boldsymbol{B}_1(\boldsymbol{r}) &= \nabla \times \boldsymbol{E}_1(\boldsymbol{r}) - \mathrm{i}\omega\mu\boldsymbol{H}_1(\boldsymbol{r}) - \mathrm{i}\omega\mu\beta\nabla \times \boldsymbol{H}_1(\boldsymbol{r}) \\ &= (\mathrm{i}\omega\mu)(\gamma/k)^2 \iiint \mathrm{d}^3\boldsymbol{r}_0\, \mathfrak{G}(\boldsymbol{r} - \boldsymbol{r}_0) \cdot [\nabla_0 \times \boldsymbol{J}(\boldsymbol{r}_0) + \beta\nabla_0 \times \nabla_0 \times \boldsymbol{J}(\boldsymbol{r}_0) \\ &\quad - \nabla_0 \times \boldsymbol{J}(\boldsymbol{r}_0) - \beta\nabla_0 \times \nabla_0 \times \boldsymbol{J}(\boldsymbol{r}_0)] = 0 \end{aligned} \tag{15a}$$

as should be the case at the source-free point $\boldsymbol{r}$. In a similar fashion, from (11*a*) and (14), it can also be shown that

$$\begin{aligned} 0 &= (\gamma/k)^2 \iiint \mathrm{d}^3\boldsymbol{r}_0\, \mathfrak{G}(\boldsymbol{r} - \boldsymbol{r}_0) \cdot [\mathfrak{D}(\boldsymbol{r}_0) \cdot \boldsymbol{J}(\boldsymbol{r}_0)] \\ &= \nabla \times \boldsymbol{E}_1(\boldsymbol{r}) - \gamma^2[\boldsymbol{E}_1(\boldsymbol{r})/\mathrm{i}\omega\mu + \beta\boldsymbol{H}_1(\boldsymbol{r})] \\ &= (\gamma/k)^2[\nabla \times \boldsymbol{H}_1(\boldsymbol{r}) + \mathrm{i}\omega\boldsymbol{D}_1(\boldsymbol{r})]. \end{aligned} \tag{15b}$$

Thus, from (15*a*, *b*) it is clear that the fields radiated by $\boldsymbol{J}$ of (11*a*) are fully compatible with Maxwell's equations.

To conclude this section, an alternative version of theorem 3 can be enunciated as follows.

Theorem 4. If there exists a magnetic current density $\boldsymbol{K}$ such that

$$\mathfrak{D} \cdot \boldsymbol{K} = 0 \tag{16a}$$

then there exists also an electric current density $\boldsymbol{J}$, given by

$$\boldsymbol{J} = -k^2(\mathrm{i}\omega\mu)^{-1}[\nabla \times \boldsymbol{K}/\gamma^2 - \beta\boldsymbol{K}] \tag{16b}$$

such that they both produce the same electric and magnetic fields.

The proof is similar to that of theorem 4.

It should be noted that by setting $\beta = 0$, the counterparts of theorems 1-4 can be easily derived for homogeneous, isotropic achiral media also. However, the application of theorems 1 and 2 differs for achiral and chiral media. Thus, while theorem 1 would be used to calculate an E-equivalent $\boldsymbol{J}$ for a given $\boldsymbol{K}$ if $\beta = 0$, it will find use in computing an E-equivalent $\boldsymbol{K}$ for a specified $\boldsymbol{J}$ when $\beta \neq 0$.

3. Duality of fields

Because of the intense use that Babinet's principle [13] finds, it is also of interest to explore the relationship of fields radiated by the electric and magnetic sources in that context. As a result, it is possible to enunciate the following duality theorem.

Theorem 5. Let $\{\boldsymbol{E}_1, \boldsymbol{H}_1\}$ be the fields produced by electric charge and current densities $\{\rho_e, \boldsymbol{J}\}$, while magnetic charge and current distributions $\{\rho_m, \boldsymbol{K}\}$ independently create the fields $\{\boldsymbol{E}_2, \boldsymbol{H}_2\}$. Then the following duality transformations hold: $\boldsymbol{E}_1 \leftrightarrow \boldsymbol{H}_2$, $\boldsymbol{H}_1 \leftrightarrow -\boldsymbol{E}_2$, $\mu \leftrightarrow \varepsilon$, $\varepsilon \leftrightarrow \mu$, while $\boldsymbol{J} \leftrightarrow \boldsymbol{K}$ and $\rho_e \leftrightarrow \rho_m$; the handedness parameter, however, $\beta \leftrightarrow \beta$.

Proof. Maxwell's equations for the fields due to the electric charge and current sources $\{\rho_e, \boldsymbol{J}\}$ are

$$\nabla \times \boldsymbol{E}_1 - \mathrm{i}\omega\boldsymbol{B}_1 = 0 \tag{17a}$$

$$\nabla \times \boldsymbol{H}_1 + \mathrm{i}\omega\boldsymbol{D}_1 = \boldsymbol{J} \tag{17b}$$

$$\nabla \cdot \boldsymbol{D}_1 - \rho_e = 0 \tag{17c}$$

$$\nabla \cdot \boldsymbol{B}_1 = 0 \tag{17d}$$

while the continuity equation is given by

$$\nabla \cdot \boldsymbol{J} - \mathrm{i}\omega\rho_e = 0. \tag{17e}$$

Similarly, it can also be shown that

$$\nabla \times \boldsymbol{H}_2 + \mathrm{i}\omega\boldsymbol{D}_2 = 0 \tag{18a}$$

$$\nabla \times \boldsymbol{E}_2 - \mathrm{i}\omega\boldsymbol{B}_2 = -\boldsymbol{K} \tag{18b}$$

$$\nabla \cdot \boldsymbol{B}_2 - \rho_m = 0 \tag{18c}$$

$$\nabla \cdot \boldsymbol{D}_2 = 0 \tag{18d}$$

$$\nabla \cdot \boldsymbol{K} - \mathrm{i}\omega\rho_m = 0. \tag{18e}$$

With the help of (1), it is then a simple matter to verify the duality of (17) and (18) via the transform given above. It ought to be pointed out that the duality transforms for achiral [14] and chiral media turn out to be identical because β remains unchanged during the transformation.

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